

ANSWERS TO KIRK-SHAHZAD'S QUESTIONS ON STRONG b -METRIC SPACES

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ABSTRACT. In this paper, two open questions on strong b -metric spaces posed by Kirk and Shahzad [13, Chapter 12] are investigated. A counter-example is constructed to give a negative answer to the first question, and a theorem on the completion of a strong b -metric space is proved to give a positive answer to the second question.

1. INTRODUCTION AND PRELIMINARIES

In 1993, Czerwik [4] introduced the notion of a b -metric which is a generalization of a metric with a view of generalizing the Banach contraction map theorem.

Definition 1.1 ([4]). Let X be a non-empty set and $d : X \times X \rightarrow [0, +\infty)$ be a function such that for all $x, y, z \in X$,

- (1) $d(x, y) = 0$ if and only if $x = y$.
- (2) $d(x, y) = d(y, x)$.
- (3) $d(x, z) \leq 2[d(x, y) + d(y, z)]$.

Then d is called a b -metric on X and (X, d) is called a b -metric space.

After that, in 1998, Czerwik [5] generalized this notion where the constant 2 was replaced by a constant $s \geq 1$, also with the name b -metric. In 2010, Khamsi and Hussain [12] reintroduced the notion of a b -metric under the name *metric-type*.

Definition 1.2 ([12], Definition 6). Let X be a non-empty set, $K > 0$ and $D : X \times X \rightarrow [0, +\infty)$ be a function such that for all $x, y, z \in X$,

- (1) $D(x, y) = 0$ if and only if $x = y$.
- (2) $D(x, y) = D(y, x)$.
- (3) $D(x, z) \leq K[D(x, y) + D(y, z)]$.

Then D is called a *metric-type* on X and (X, D, K) is called a *metric-type space*.

Definition 1.3 ([12], Definition 7). Let (X, D, K) be a b -metric space.

- (1) A sequence $\{x_n\}$ is called *convergent* to x , written as $\lim_{n \rightarrow \infty} x_n = x$, if $\lim_{n \rightarrow \infty} D(x_n, x) = 0$.
- (2) A sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n, m \rightarrow \infty} D(x_n, x_m) = 0$.
- (3) (X, D, K) is called *complete* if every Cauchy sequence is a convergent sequence.

From Definition 1.2.(3), it is easy to see that $K \geq 1$. Also in 2010, Khamsi [11] introduced another definition of a metric-type where the condition (3) in Definition 1.2 was replaced by

$$D(x, z) \leq K[D(x, y_1) + \dots + D(y_n, z)]$$

for all $x, y_1, \dots, y_n, z \in X$, see [11, Definition 2.7]. In the sequel, the metric-type in the sense of Khamsi and Hussain [12] will be called a b -metric to avoid the confusion about the metric-type in the sense of Khamsi [11]. Note that every metric-type is a b -metric.

The same relaxation of the triangle inequality in Definition 1.2 was also discussed in 2003 by Fagin *et al.* [8], who called this new distance measure nonlinear elastic matching. The authors of that paper remarked that this measure had been used, for example, in [9] for trademark shapes and in [3] to measure ice floes. In 2009, Xia [15] used this semimetric distance to study the optimal transport path between probability measures.

In recent times, b -metric spaces were studied by many authors, especially fixed point theory on b -metric spaces [1], [7], [10], [13, Chapter 12], [14]. Some authors were also studied topological properties of b -metric spaces. In [2], An *et al.* showed that every b -metric space with the topology induced by its convergence is a semi-metrizable space and thus many properties of b -metric spaces used in the literature are obvious. Then, the authors proved the Stone-type theorem on b -metric spaces and get a sufficient condition for a b -metric space to be metrizable. Notice that a b -metric space is always understood to be a topological space with respect to the topology induced by its convergence and a b -metric need not be continuous [2, Examples 3.9 & 3.10]. This fact suggests a strengthening of the notion of b -metric spaces which remedies this defect.

Definition 1.4 ([13], Definition 12.7). Let X be a non-empty set, $K \geq 1$ and $D : X \times X \rightarrow [0, +\infty)$ be a function such that for all $x, y, z \in X$,

- (1) $D(x, y) = 0$ if and only if $x = y$.
- (2) $D(x, y) = D(y, x)$.
- (3) $D(x, z) \leq D(x, y) + KD(y, z)$.

Then D is called a *strong b -metric* on X and (X, D, K) is called a *strong b -metric space*.

Remark 1.5 ([13], page 122). (1) Every strong b -metric is continuous.

- (2) Every open ball $B(a, r) = \{x \in X : D(a, x) < r\}$ of a strong b -metric space (X, D, K) is open.

In [13, Chapter 12], Kirk and Shahzad surveyed b -metric spaces, strong b -metric spaces, and related problems. An interesting work was attracted many authors is to transform results of metric spaces to the setting of b -metric spaces. It is only fair to point out that some results seem to require the full use of the triangle inequality of a metric space. In this connection, Kirk and Shahzad [13, page 127] mentioned an interesting extension of Nadler's theorem due to Dontchev and Hager [6]. Recall that for a metric space (X, d) and $A, B \subset X$, $x \in X$,

$$\text{dist}(x, A) = \inf\{d(x, a) : a \in A\}$$

$$\delta(A, B) = \sup\{\text{dist}(x, A) : x \in B\}$$

and these notation are understood similarly on b -metric spaces.

Theorem 1.6 ([13], Theorem 12.7). Let (X, d) be a complete metric space, $T : X \rightarrow X$ be a map from X into a non-empty closed subset of X , and $x_0 \in X$ such that

- (1) $\text{dist}(x_0, Tx_0) < r(1 - k)$ for some $r > 0$ and some $k \in [0, 1)$.
- (2) $\delta(Tx \cap B(x_0, r), Ty) \leq kd(x, y)$ for all $x, y \in B(x_0, r)$.

Then T has a fixed point in $B(x_0, r)$.

Based on the definition of $\delta(A, B)$ and the proof of Theorem [13, Theorem 12.7], the assumption (2) in the above theorem is implicitly understood as

$\delta(Tx \cap B(x_0, r), Ty) \leq kd(x, y)$ for all $x, y \in B(x_0, r)$ and $Tx \cap B(x_0, r) \neq \emptyset$.

The authors of [13] did not know whether Theorem 1.6 holds under the weaker strong b -metric assumption. Explicitly, we have the following question.

Question 1.7 ([13], page 128). *Let (X, D, K) be a complete strong b -metric space, $T : X \rightarrow X$ be a map from X into a non-empty closed subset of X , and $x_0 \in X$ such that*

- (1) $\text{dist}(x_0, Tx_0) < r(1 - k)$ for some $r > 0$ and some $k \in [0, 1)$.
- (2) $\delta(Tx \cap B(x_0, r), Ty) \leq kD(x, y)$ for all $x, y \in B(x_0, r)$ and $Tx \cap B(x_0, r) \neq \emptyset$.

Does the map T have a fixed point in $B(x_0, r)$?

Recall that a map $f : X \rightarrow Y$ from a b -metric space (X, D, K) into a b -metric space (Y, D', K') is called an *isometry* if $D'(f(x), f(y)) = D(x, y)$ for all $x, y \in X$. Also, a b -metric space (X^*, D^*, K^*) is called a *completion* of the b -metric space (X, D, K) if (X^*, D^*, K^*) is complete and there exists an isometry $f : X \rightarrow X^*$ such that $\overline{f(X)} = X^*$. A classical result is that every metric space is dense in a complete metric space. So, it is interesting to ask whether this result holds or not in the setting of strong b -metric spaces.

Question 1.8 ([13], page 128). *Is every strong b -metric space dense in a complete strong b -metric space?*

Kirk and Shahzad [13, page 128] commented that if the answer of Question 1.8 is positive, then every contraction map $f : X \rightarrow X$ on a strong b -metric space X may be extended to a contraction map $f : X^* \rightarrow X^*$ on a complete strong b -metric space X^* which has a unique fixed point. Ostrowski's theorem [13, Theorem 12.6] then would provide a method for approximating this fixed point.

In this paper, two above questions on strong b -metric spaces are investigated. A counter-example is constructed to give a negative answer to Question 1.7, and a theorem on the completion of a strong b -metric space is proved to give a positive answer to Question 1.8.

2. MAIN RESULTS

First, the following example gives a negative answer to Question 1.7.

Example 2.1. Let $X = \{1, 2, 3\}$, $D : X \times X \rightarrow [0, +\infty)$ be defined by $D(1, 1) = D(2, 2) = D(3, 3) = 0$, $D(1, 2) = D(2, 1) = 2$, $D(2, 3) = D(3, 2) = 1$, $D(1, 3) = D(3, 1) = 6$ and a map $T : X \rightarrow X$ be defined by

$$T1 = 2, T2 = 3, T3 = 1.$$

Then

- (1) (X, D, K) is a complete strong b -metric space with $K = 4$.
- (2) T and (X, D, K) satisfy all assumptions of Question 1.7 with $x_0 = 1$, $r = 6$, $k = \frac{1}{2}$.
- (3) T has no any fixed point.

Proof. (1). For all $x, y \in X$, it follows from definition of D that $D(x, y) = D(y, x)$ and

$$D(x, y) = 0 \text{ if and only if } x = y.$$

We also have

$$D(1, 3) + KD(3, 2) = 6 + 4 \cdot 1 = 10 \geq 2 = D(1, 2)$$

$$\begin{aligned}
KD(1, 3) + D(3, 2) &= 4.6 + 1 = 25 \geq 2 = D(1, 2) \\
D(1, 2) + KD(2, 3) &= 2 + 4.1 = 6 = D(1, 3) \\
KD(1, 2) + D(2, 3) &= 4.2 + 1 = 9 \geq 6 = D(1, 3) \\
D(2, 1) + KD(1, 3) &= 2 + 4.6 = 26 \geq 1 = D(2, 3) \\
KD(2, 1) + D(1, 3) &= 4.2 + 6 = 14 \geq 1 = D(2, 3).
\end{aligned}$$

By the above, D is a strong b -metric on X . Since X is finite and discrete, X is complete. So, (X, D, K) is a complete strong b -metric space with $K = 4$.

(2). Since $TX = X$, TX is a non-empty closed subset of X . We have

$$\text{dist}(x_0, Tx_0) = \text{dist}(1, T1) = \text{dist}(1, \{2\}) = D(1, 2) = 2$$

and

$$r(1 - k) = 6\left(1 - \frac{1}{2}\right) = 3.$$

This proves that $\text{dist}(x_0, Tx_0) < r(1 - k)$.

We also have $B(x_0, r) = B(1, 6) = \{1, 2\}$.

If $x = y = 1$, then $Tx \cap B(x_0, r) = \{2\}$ and

$$\delta(Tx \cap B(x_0, r), Ty) = \delta(\{2\}, \{2\}) = D(2, 2) = 0 \leq kD(x, y).$$

If $x = y = 2$, then $Tx \cap B(x_0, r) = \emptyset$.

If $x = 1, y = 2$, then

$$\delta(Tx \cap B(x_0, r), Ty) = \delta(\{2\}, \{3\}) = D(2, 3) = 1 = \frac{1}{2}D(1, 2) = kD(x, y).$$

If $x = 2, y = 1$, then $Tx \cap B(x_0, r) = \emptyset$.

By the above, $\delta(Tx \cap B(x_0, r), Ty) \leq kD(x, y)$ for all $x, y \in B(x_0, r)$ and $Tx \cap B(x_0, r) \neq \emptyset$.

(3). By definition of T , we see that T has no any fixed point. \square

Next, the following theorem is a positive answer to Question 1.8.

Theorem 2.2. *Let (X, D, K) be a strong b -metric space. Then*

- (1) (X, D, K) has a completion (X^*, D^*, K) .
- (2) The completion of (X, D, K) is unique in the sense that if (X_1^*, D_1^*, K_1) and (X_2^*, D_2^*, K_2) are two completions of (X, D, K) , then there is a bijective isometry $\varphi : X_1^* \rightarrow X_2^*$ which restricts to the identity on X .

Proof. Put

$$\mathcal{C} = \{\{x_n\} : \{x_n\} \text{ is a Cauchy sequence in } (X, D, K)\}.$$

Define a relation \sim on \mathcal{C} as follows:

$$\{x_n\} \sim \{y_n\} \text{ if and only if } \lim_{n \rightarrow \infty} D(x_n, y_n) = 0, \text{ for all } \{x_n\}, \{y_n\} \in \mathcal{C}.$$

The relation \sim obviously satisfies reflexivity and symmetry. If $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$, then $\lim_{n \rightarrow \infty} D(x_n, y_n) = \lim_{n \rightarrow \infty} D(y_n, z_n) = 0$. Since

$$0 \leq D(x_n, z_n) \leq D(x_n, y_n) + KD(y_n, z_n)$$

for all n , $\lim_{n \rightarrow \infty} D(x_n, z_n) = 0$. Thus $\{x_n\} \sim \{z_n\}$. Therefore, the relation \sim is an equivalent relation on \mathcal{C} .

Denote

$$X^* = \{x^* = [\{x_n\}] : \{x_n\} \in \mathcal{C}\}$$

where $x^* = [\{x_n\}]$ is an equivalence class of $\{x_n\}$ under the relation \sim , and define a function $D^* : X^* \times X^* \rightarrow \mathbb{R}$ by

$$D^*(x^*, y^*) = \lim_{n \rightarrow \infty} D(x_n, y_n). \quad (2.1)$$

We see that, for all n, m

$$\begin{aligned} D(x_n, y_n) &\leq KD(x_n, x_m) + D(x_m, y_n) \\ &\leq KD(x_n, x_m) + D(x_m, y_m) + KD(y_m, y_n). \end{aligned}$$

It implies that

$$D(x_n, y_n) - D(x_m, y_m) \leq K[D(x_n, x_m) + D(y_m, y_n)]. \quad (2.2)$$

Also

$$\begin{aligned} D(x_m, y_m) &\leq KD(x_m, x_n) + D(x_n, y_n) \\ &\leq KD(x_n, x_m) + D(x_n, y_n) + KD(y_n, y_m). \end{aligned}$$

Therefore,

$$D(x_m, y_m) - D(x_n, y_n) \leq K[D(x_n, x_m) + D(y_m, y_n)]. \quad (2.3)$$

It follows from (2.2) and (2.3) that

$$|D(x_m, y_m) - D(x_n, y_n)| \leq K[D(x_n, x_m) + D(y_m, y_n)]. \quad (2.4)$$

Taking the limit as $n, m \rightarrow \infty$ in (2.4), we get $\lim_{n, m \rightarrow \infty} |D(x_m, y_m) - D(x_n, y_n)| = 0$, that is, $\{D(x_n, y_n)\}$ is a Cauchy sequence in \mathbb{R} . Thus $\lim_{n \rightarrow \infty} D(x_n, y_n)$ exists.

Moreover, if $\{x_n\} \sim \{z_n\}$ and $\{y_n\} \sim \{w_n\}$, then

$$\lim_{n \rightarrow \infty} D(x_n, z_n) = \lim_{n \rightarrow \infty} D(y_n, w_n) = 0. \quad (2.5)$$

We see that

$$\begin{aligned} D(x_n, y_n) &\leq KD(x_n, z_n) + D(z_n, y_n) \\ &\leq KD(x_n, z_n) + D(z_n, w_n) + KD(w_n, y_n). \end{aligned}$$

It implies that

$$D(x_n, y_n) - D(z_n, w_n) \leq KD(x_n, z_n) + KD(w_n, y_n).$$

Similarly,

$$D(z_n, w_n) - D(x_n, y_n) \leq KD(z_n, x_n) + KD(y_n, w_n).$$

Therefore,

$$|D(x_n, y_n) - D(z_n, w_n)| \leq KD(x_n, z_n) + KD(w_n, y_n). \quad (2.6)$$

Taking the limit as $n, m \rightarrow \infty$ in (2.6) and using (2.5), we get $\lim_{n \rightarrow \infty} |D(x_n, y_n) - D(z_n, w_n)| = 0$. Thus $\lim_{n \rightarrow \infty} D(x_n, y_n) = \lim_{n \rightarrow \infty} D(z_n, w_n)$. Therefore, the function D^* is well-defined.

In the next, we shall prove that (X^*, D^*, K) is a strong b -metric space. For all $x^*, y^*, z^* \in X^*$, we have

$$D^*(x^*, y^*) = \lim_{n \rightarrow \infty} D(x_n, y_n) \geq 0 \text{ since } D(x_n, y_n) \geq 0 \text{ for all } n.$$

$D^*(x^*, y^*) = 0$ if and only if $\lim_{n \rightarrow \infty} D(x_n, y_n) = 0$, that is, $\{x_n\} \sim \{y_n\}$. It is equivalent to $x^* = y^*$.

$D^*(x^*, y^*) = \lim_{n \rightarrow \infty} D(x_n, y_n) = \lim_{n \rightarrow \infty} D(y_n, x_n) = D^*(y^*, x^*)$ since $D(x_n, y_n) = D(y_n, x_n)$ for all n .

$$D^*(x^*, z^*) = \lim_{n \rightarrow \infty} D(x_n, z_n) \leq \lim_{n \rightarrow \infty} [D(x_n, y_n) + KD(y_n, z_n)] = D^*(x^*, y^*) + KD^*(y^*, z^*).$$

So, (X^*, D^*, K) is a strong b -metric space.

For each $x \in X$, put $f(x) = [\{x, x, x, \dots\}] \in X^*$. We see that f is an isometry from (X, D, K) into (X^*, D^*, K) since

$$D^*(f(x), f(y)) = \lim_{n \rightarrow \infty} D(x, y) = D(x, y)$$

for all $x, y \in X$.

Next, we will prove that $f(X)$ is dense in X^* . If $x^* = [\{x_n\}] \in X^*$, then $\lim_{n, m \rightarrow \infty} D(x_n, x_m) = 0$.

For each $i \in \mathbb{N}$, there exists n_0^i such that $D(x_n, x_m) \leq \frac{1}{i}$ for all $n, m \geq n_0^i$. It implies that

$$0 \leq D^*(f(x_{n_0^i}), x^*) = \lim_{n \rightarrow \infty} D(x_{n_0^i}, x_n) \leq \frac{1}{i}.$$

So $\lim_{i \rightarrow \infty} D^*(f(x_{n_0^i}), x^*) = 0$. This proves that $\lim_{i \rightarrow \infty} f(x_{n_0^i}) = x^*$, that is, $f(X)$ is dense in X^* .

Next, we will prove that (X^*, D^*, K) is complete. Let $\{x_n^*\}$ be a Cauchy sequence in X^* , where $x_n^* = [\{x_i^n\}_i]$ for some $\{x_i^n\}_i \in \mathcal{C}$. Then

$$\lim_{n, m \rightarrow \infty} D^*(x_n^*, x_m^*) = 0. \quad (2.7)$$

Note that the open ball $B(x_n^*, \frac{1}{Kn})$ is open by Remark 1.5.(2). From the fact that $f(X)$ is dense in X^* , for each n there exists $y_n \in X$ such that

$$D^*(f(y_n), x_n^*) < \frac{1}{Kn}. \quad (2.8)$$

By (2.8), for all n, m , we have

$$\begin{aligned} D(y_n, y_m) &= D^*(f(y_n), f(y_m)) \\ &\leq KD^*(f(y_n), x_n^*) + D^*(x_n^*, f(y_m)) \\ &\leq KD^*(f(y_n), x_n^*) + D^*(x_n^*, x_m^*) + KD^*(x_m^*, f(y_m)) \\ &< \frac{1}{n} + D^*(x_n^*, x_m^*) + \frac{1}{m}. \end{aligned} \quad (2.9)$$

Taking the limit as $n, m \rightarrow \infty$ in (2.9) and using (2.7), we get

$$\lim_{n, m \rightarrow \infty} D(y_n, y_m) = 0. \quad (2.10)$$

Thus $\{y_n\}$ is a Cauchy sequence in (X, D, K) . Put $y^* = [\{y_n\}] \in X^*$. From (2.8), we have

$$\begin{aligned} D^*(x_n^*, y^*) &\leq KD^*(x_n^*, f(y_n)) + D^*(f(y_n), y^*) \\ &< K \frac{1}{Kn} + \lim_{m \rightarrow \infty} D(y_n, y_m) \\ &= \frac{1}{n} + \lim_{m \rightarrow \infty} D(y_n, y_m). \end{aligned} \quad (2.11)$$

Taking the limit as $n \rightarrow \infty$ in (2.11) and using (2.10), we have $\lim_{n \rightarrow \infty} D^*(x_n^*, y^*) = 0$, that is, $\lim_{n \rightarrow \infty} x_n^* = y^*$ in (X^*, D^*, K) . Therefore, (X^*, D^*, K) is complete.

Finally, we prove the uniqueness of the completion. Let (X_1^*, D_1^*, K_1) and (X_2^*, D_2^*, K_2) be two completions of (X, D, K) . For each $x_1^* \in X_1^*$, there exists $\{x_n\} \subset X$ such that $\lim_{n \rightarrow \infty} f_1(x_n) = x_1^*$ where $f_1 : X \rightarrow X_1^*$ is an isometry. Since $\{f_1(x_n)\}$ is convergent, $\{f_1(x_n)\}$ is a Cauchy sequence in X_1^* . Since f_1 is an isometry, $\{x_n\}$ is a Cauchy sequence in X . Note that there exists $f_2 : X \rightarrow X_2^*$ which is also an isometry. Then $\{f_2(x_n)\}$ is a Cauchy sequence in X_2^* and thus there exists $x_2^* \in X_2^*$ such that $\lim_{n \rightarrow \infty} f_2(x_n) = x_2^*$. Define $\varphi : X_1^* \rightarrow X_2^*$ by $\varphi(x_1^*) = x_2^*$.

If $y_2^* \in X_2^*$, then $y_2^* = \lim_{n \rightarrow \infty} f_2(y_n)$ for some $\{y_n\} \subset X$. Since $\{f_2(y_n)\}$ is convergent, $\{f_2(y_n)\}$ is a Cauchy sequence in X_2^* . Since f_2 is an isometry, $\{y_n\}$ is a Cauchy sequence in X . Also, f_1 is an isometry, $\{f_1(y_n)\}$ is a Cauchy sequence in X_1^* . Then there exists $y_1^* = \lim_{n \rightarrow \infty} f_1(y_n)$. Therefore, $y_2^* = \varphi(y_1^*)$. This proves that φ is bijective. Moreover, for every $x^*, y^* \in X_1^*$ with $x^* = \lim_{n \rightarrow \infty} f_1(x_n)$ and $y^* = \lim_{n \rightarrow \infty} f_1(y_n)$, by using the continuity of D_1^* and D_2^* , we have

$$\begin{aligned} D_1^*(x^*, y^*) &= \lim_{n \rightarrow \infty} D_1^*(f_1(x_n), f_1(y_n)) = \lim_{n \rightarrow \infty} D(x_n, y_n) \\ &= \lim_{n \rightarrow \infty} D_2^*(f_2(x_n), f_2(y_n)) = D_2^*(\varphi(x^*), \varphi(y^*)). \end{aligned}$$

It implies that φ is a bijective isometry $\varphi : X_1^* \rightarrow X_2^*$ which restricts to the identity on X . \square

Finally, the following example shows that techniques used in the proof of Theorem 2.2 may not be applied to b -metric spaces.

Example 2.3. Let $X = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$ and

$$D(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0, 1\} \\ |x - y| & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n = 1, 2, \dots\} \\ 4 & \text{otherwise.} \end{cases}$$

Then D is a b -metric on X with $K = \frac{8}{3}$ [2, Example 3.9]. Put $x_n = 1$, $y_n = \frac{1}{2n}$, $z_n = 1$ and $w_n = 0$ for all n . Then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$ are Cauchy sequences and $\{x_n\} \sim \{z_n\}$ and $\{y_n\} \sim \{w_n\}$. However,

$$\lim_{n \rightarrow \infty} D(x_n, y_n) = \lim_{n \rightarrow \infty} D(1, \frac{1}{2n}) = 4 \neq 1 = D(1, 0) = \lim_{n \rightarrow \infty} D(z_n, w_n).$$

This shows that the formula (2.1) is not well-defined for the above b -metric D .

Though the above example shows that that techniques used in the proof of Theorem 2.2 may not be applied to b -metric spaces, we do not know whether Theorem 2.2 fully extends to b -metric spaces. So, we conclude with the following question.

Question 2.4. *Does every b -metric space have a completion?*

REFERENCES

- [1] T. V. An, N. V. Dung, Z. Kadelburg, and S. Radenović, *Various generalizations of metric spaces and fixed point theorems*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **109** (2015), 175 – 198.
- [2] T. V. An, L. Q. Tuyen, and N. V. Dung, *Stone-type theorem on b -metric spaces and applications*, Topology Appl. **185 - 186** (2015), 50 – 64.
- [3] R. M. Connell, R. Kwok, J. Curlander, W. Kober, and S. Pang, *Ψ - S correlation and dynamic time warping: two methods for tracking ice floes in SAR images*, IEEE Trans. Geosci. Remote Sens. **29** (1991), no. 6, 1004 – 1012.
- [4] S. Czerwik, *Contraction mappings in b -metric spaces*, Acta Math. Univ. Ostrav. **1** (1993), no. 1, 5 – 11.

- [5] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Math. Fis. Univ. Modena **46** (1998), 263 – 276.
- [6] A. L. Dontchev and W. W. Hager, *An inverse mapping theorem for set-valued maps*, Proc. Amer. Math. Soc. **121** (1994), no. 2, 481 – 489.
- [7] N. V. Dung, N. T. T. Ly, V. D. Thinh, and N. T. Hieu, *Suzuki-type fixed point theorems for two maps in metric-type spaces*, J. Nonlinear Anal. Optim. **4** (2013), no. 2, 17 – 29.
- [8] R. Fagin, R. Kumar, and D. Sivakumar, *Comparing top k lists*, Siam J. Discrete Math. **17** (2003), no. 1, 134 – 160.
- [9] R. Fagin and L. Stockmeyer, *Relaxing the triangle inequality in pattern matching*, Int. J. Comput. Vis. **30** (1998), no. 3, 219 – 231.
- [10] N. T. Hieu and N. V. Dung, *Some fixed point results for generalized rational type contraction mappings in partially ordered b-metric spaces*, Facta Univ. Ser. Math. Inform. **30** (2015), no. 1, 47 – 64.
- [11] M. A. Khamsi, *Remarks on cone metric spaces and fixed point theorems of contractive mappings*, Fixed Point Theory Appl. **2010** (2010), 1 – 7.
- [12] M. A. Khamsi and N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal. **7** (2010), no. 9, 3123 – 3129.
- [13] W. Kirk and N. Shahzad, *Fixed point theory in distance spaces*, Springer, 2014.
- [14] P. Kumam, N. V. Dung, and V. T. L. Hang, *Some equivalences between cone b-metric spaces and b-metric spaces*, Abstr. Appl. Anal. **2013** (2013), 1 – 8.
- [15] Q. Xia, *The geodesic problem in quasimetric spaces*, J. Geom. Anal. **19** (2009), 452 – 479.

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